This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE ICC 2010 proceedings

Duality-Gap Bounds for Multi-Carrier Systems and Their Application to Periodic Scheduling

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***Abstract*—We investigate a novel cross-layer optimization problem for jointly performing dynamic spectrum management (DSM) and periodic rate-scheduling in time. The large number of carriers used in digital subscriber lines (DSL) makes DSM a large-scale optimization problem for which dual optimization is a commonly used method. The duality-gap which potentially accompanies the dual optimization for non-convex problems is typically assumed to be small enough to be neglected. Also, previous theoretical results show a vanishing duality-gap as the number of subcarriers approaches infinity.**

We will bound the potential performance improvements that can be achieved by the additional rate-scheduling procedure. This bound is found to depend on the duality-gap in the physical layer DSM problem. Furthermore, we will derive bounds on the duality-gap of the two most important optimization problems in DSL, namely the maximization of the weighted sum-rate and the minimization of the weighted sum-power. These bounds are derived for a finite number of subcarriers and are also applicable to the respective problems in orthogonal frequency division multiplex (OFDM) systems.

I. Introduction

In this paper we study the cross-layer problem of optimizing the power spectrum in multi-carrier systems jointly with the target-rate allocation in time. The goal of the latter is interference avoidance for reducing the average power consumption irrespective of the link congestion. Assuming the channel is slowly varying, periodically repeated finite-length rate-schedules can be precomputed together with the respective optimal power spectrum allocation and only updated when channel conditions change. Hence, this idea gives performance improvements at lower implementation complexity compared to *optimal* traffic-dependent cross-layer schedulers. While we focus in our derivation on power allocation problems in digital subscriber lines (DSL), also known as dynamic spectrum management (DSM) problems, all results can be straightforwardly applied to the respective orthogonal frequency division multiplex (OFDM) problems [1].

While in [2] a heuristic approach was used to avoid interference by joint DSM and user-allocation in time, in [3] the decomposable structure of the mathematical program was exploited to arrive at a *layered* algorithm structure, *cf.* also the related interference network problem in [4]. However, in [3] it was demonstrated by a practical simulation example that periodic scheduling yields negligible additional energy savings compared to optimal DSM. Similarly, *continuous* time-sharing of subcarriers among users has been used in the OFDM literature as a convex relaxation of non-convex resource allocation problems [5].

We will show that our rate-scheduling problem is equivalent to a practical implementation of this approximation where subcarriers are periodically allocated to users for a fraction of the OFDM symbols. Furthermore, we will derive a *duality-gap*-dependent bound on the energy savings by periodic scheduling. Thereby we analytically explain the observations in [3] on the practical potential of periodic scheduling and simultaneously investigate the impact of the mentioned time-sharing relaxation in [5]. Differently to previous work [1] we will demonstrate the *target-rate* dependency of the duality-gap in non-convex multi-carrier power allocation problems. Furthermore, we derive bounds on this duality-gap in the weighted sum-power minimization and weighted sum-rate maximization problem, respectively, which in contrast to those in [6], [7] are applicable for a finite number of subcarriers.

This paper is organized as follows. In Section II we introduce the system model and the two considered DSM optimization problems. Next we define and analyze the cross-layer scheduling problem in Section III. The duality-gap in DSM problems is then the topic of Section IV which concludes with bounds thereof. After showing simulation results comparing the different bounds in Section V, in Section VI we draw conclusions from this work.

II. Model and DSM Problems

A. System Model

We regard an interference-limited DSL scenario with U subscriber lines, employing discrete multi-tone (DMT) modulation and spectrum level coordination. We will assume perfect duplexing, synchronization among modems, and access to the magnitudes of all crosstalk couplings at least at the collocated side. However, the extension of this work to the full duplex case incorporating near-end crosstalk is straightforward. The C subcarriers can hence be modeled as orthogonal subchannels and one obtains a far-end crosstalk limited system. In the following users and subcarriers are identified by the sets of indices $\mathcal{U} = \{1, \ldots, U\}$ and $\mathcal{C} = \{1, \ldots, C\}$,

This work has been supported in parts by the Austrian Government and the City of Vienna within the competence center program COMET.

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respectively. Under Gaussian-noise approximation and two-dimensional signal constellations the achievable rate per DMT-symbol for user $u \in \mathcal{U}$ on subcarrier $c \in \mathcal{C}$ is given by [8]

$$r_c^u(\mathbf{p}_c) = \log_2\left(1 + \frac{H_c^{uu} p_c^u}{\Gamma(\sum_{i\in\mathcal{U}\setminus u} H_c^{ui} p_c^i \ + \ N_c^u)}\right), \quad (1)$$

where $\mathbf{p}_c = [p_c^1, \dots, p_c^U]^T$ and p_c^u is the power spectral density on subcarrier c of user u. We denote the squared magnitudes of the direct channel transfer coefficient of user u by H_c^{uu}, and that of the cross-channel transfer coefficient from user i to user u by H_c^{ui}, respectively. We further utilize the SNR-gap to capacity Γ and write the total background noise power spectral density on subcarrier c and line u as N_c^u.

B. Physical Layer DSM Problems

We pose the multi-user DSM problem of minimizing the weighted sum-power in DSL as

$$\begin{aligned} P^*(\mathbf{R}) \ = \ \underset{\mathbf{p}}{\text{minimize}} \ & \sum_{u\in\mathcal{U}} w_u \sum_{c\in\mathcal{C}} \ p_c^u & (2a)\\ \text{subject to} \ & \sum_{c\in\mathcal{C}} r_c^u(\mathbf{p}_c) \ \geq \ R_u, \quad \forall u \in \mathcal{U}, & (2b)\\ & \mathbf{p}_c \in \mathcal{Q}_c, \qquad \forall c \in \mathcal{C}, & (2c)\end{aligned}$$

where $\mathbf{p} \ = \ [(\mathbf{p}_1)^T, \dots, (\mathbf{p}_C)^T]^T$ and

$$\mathcal{Q}_c = \{\mathbf{p}_c | r_c^u(\mathbf{p}_c) \in \mathcal{B}_c^u, \ 0 \leq p_c^u \leq \hat{p}_c^u, \forall u \in \mathcal{U}\} \quad (3)$$

is the set of feasible PSD's on subcarrier c. Therein $\mathcal{B}_c^u \subset \mathcal{Z}_+$ is the set of positive, discrete bit-allocations per subcarrier, potentially additionally bounded by a restriction on the maximal number of bits that may be loaded on this subcarrier, and we assume a spectral mask constraint $\hat{p}_c^u$, $\forall u \in \mathcal{U}, c \in \mathcal{C}$. For generality we include per-user weights $w_u, u \in \mathcal{U}$, restricted to the open simplex $w_u > 0$, $\sum_{u\in\mathcal{U}} w_u = 1$, allowing us to trace the boundary of the sum-power tradeoff curve or to include higher-layer information [5]. Finally, $\mathbf{R} \in \mathcal{R}_+^U$ are the integer target-rates of all users in [bits/DMT-symbol]. The total transmit energy $f_s^{-1} \sum_{u\in\mathcal{U}} \sum_{c\in\mathcal{C}} \ p_c^u$, where f_s denotes the DMT-symbol frequency, is only a scaled version of our objective and hence minimized by the optimum of (2).

One may regard (2) as complementary to the standard DSM problem formulation for maximizing the weighted sum-rate, which is given as

$$\begin{aligned} \underset{\mathbf{p}}{\text{maximize}} \ & \sum_{u\in\mathcal{U}} w_u \sum_{c\in\mathcal{C}} \ r_c^u(\mathbf{p}_c) & (4a)\\ \text{subject to} \ & \sum_{c\in\mathcal{C}} p_c^u \ \leq \ P_u^{\max}, \qquad \forall u \in \mathcal{U}, & (4b)\\ & \mathbf{p}_c \in \mathcal{Q}_c, \qquad \forall c \in \mathcal{C}, & (4c)\end{aligned}$$

where $\mathbf{P}^{\max} \in \mathcal{R}^U$ denotes the maximum transmit power of all users $u \in \mathcal{U}$. Both optimization problems (2) and (4) are noncontinuous and nonconvex, at least when U and C are both greater than one. A difference between them is however that while the latter is always feasible, (2) does not necessarily have a feasible solution.

In the remaining of this section we outline the partial dual problems to primal problems (2) and (4). The gap in objective value between primal and dual optimal solutions will be bounded in Section IV, while in the next section it will be used to bound the performance improvement by additionally including periodic scheduling in our optimization. Dual relaxation, beside its algorithmic applications, has been applied for the decomposition of various problems in communications, so also for DSM [9] where constraints (2b) and (4b), respectively, couple the per-carrier power allocations. The convex, partial dual problem to (2) is given as

$$D^*(\mathbf{R}) \ = \ \underset{\boldsymbol{\lambda}\succeq\mathbf{0}}{\text{maximize}} \ \ q(\boldsymbol{\lambda}), \quad (5)$$

where the concave dual function is defined as

$$q(\boldsymbol{\lambda}) = \min_{\{\mathbf{p}|\mathbf{p}_c\in\mathcal{Q}_c,\forall c\in\mathcal{C}\}} \ \Big\{\sum_{c\in\mathcal{C}} L_c(\mathbf{p}_c, \boldsymbol{\lambda})\Big\}, \quad (6)$$

the partial Lagrangian is written as a sum of per-carrier Lagrangians

$$L_c(\mathbf{p}_c, \boldsymbol{\lambda}) \ = \ \sum_{u\in\mathcal{U}} w_u \ p_c^u + \lambda_u \left(\frac{R_u}{C} - r_c^u(\mathbf{p}_c)\right), \quad (7)$$

and $\boldsymbol{\lambda} \in \mathcal{R}^U$ are the Lagrange multipliers associated with the relaxed *coupling* constraints (2b). Note that $r_c^u(\mathbf{p}_c)$ is a quasi-convex function[1], giving the primal, per-carrier minimization subproblems in (6) the interpretation of minimizing a sum of quasi-convex functions. This, beside the bit-loading, can be seen as the origin of the nonconvexity of this problem which persists even in the continuous bit-loading case. The partial dual problem to (4) can be constructed in a similar way as (5) by relaxing the sum-power constraints (4b).

While in Problem (2) we have omitted per-user sum-power constraints as in (4b), all forthcoming results and conclusions hold also in the case where such additional constraints are present. Dual Problem (5) would change as it would also include relaxation terms for these constraints in the objective and the corresponding Lagrange multipliers as additional variables. Also note that both primal problems can easily be adapted to capture orthogonal allocation constraints typically considered in the OFDM literature [1], [5]. This can be accomplished by adding the constraint

$$p_c^u p_c^i = 0, \forall i \in \mathcal{U} \setminus u, \forall u \in \mathcal{U}, \quad (8)$$

in problems (2) and (4). Remarkably all of the following derivations remain valid even in this case.

It is repeatedly assumed in literature that dual optimal DSM algorithms reach near-optimal solutions. However, theoretically [11] there remains a possibly non-zero gap

$$\zeta \doteq P^*(\mathbf{R}) - D^*(\mathbf{R}) \quad (9)$$

between the optimal cost $P^*(\mathbf{R})$ of the primal problem (2) (or (4)) and the optimal dual cost $D^*(\mathbf{R})$ to (5) (or the mentioned

[1]This follows from quasi-convexity of the logarithm's argument in (1) [10, Ex. 3.32] and composition with the (nondecreasing) logarithm [10, Sec. 3.4.4].

partial dual to (4)), respectively, due to the nonconvexity of the constraints in (2) and (4), respectively. After discussing the idea of periodic scheduling in the following section, in section IV we will return to this topic and investigate bounds for ζ.

III. Periodic Scheduling

In Problem (2) the users' target rates were fixed. In periodic scheduling we aim for a finite-length rate-schedule where target rates are allowed to vary but average to the original target rates $R_u,\ u \in \mathcal{U}$. This schedule is assumed to be repeated periodically in time and only updated when channel conditions change. Although traffic variability cannot be exploited by this strategy, it allows for energy reduction by interference avoidance in the time domain in addition to that in the frequency domain. While target-rate constraints are supposedly chosen in accordance with a sevice-level agreement, sum-power constraints as in (4b) are in fact fixed engineering constraints. The periodic scheduling strategy can therefore not be applied in conjunction with Problem (4).

We will index the DMT-symbols in the schedule of length N by $\mathcal{N} = \{1, \dots, N\}$. Furthermore, we denote the vector of scheduling variables (*i.e.*, rates assigned to symbols) by $\mathbf{d} = [(\mathbf{d}^1)^T, \dots, (\mathbf{d}^U)^T]^T$, where $\mathbf{d}^u = [d^{u,(1)}, \dots, d^{u,(N)}]^T$ is the vector of target-rates assigned by user u to all symbols $n \in \mathcal{N}$, and $\mathbf{d}^{(n)} = [d^{1,(n)}, \dots, d^{U,(n)}]^T$. Similarly we may extend the notation of power-allocations $\mathbf{p}$ to multiple symbols, *e.g.*, by denoting the PSD of user u on subcarrier c and symbol n by $p_c^{u,(n)}$.

Using $P^*(\mathbf{d}^{(n)})$, the optimum objective value of (2), we write the cross-layer problem of jointly performing DSM and periodic scheduling as

$$P_S^* \;=\; \underset{\mathbf{d} \in \mathcal{D}}{\text{minimize}} \quad \sum_{n \in \mathcal{N}} P^*(\mathbf{d}^{(n)}) \tag{10}$$

where

$$\mathcal{D} = \{\mathbf{d} | \mathbf{d} \in \mathcal{Z}_+^{UN}, \sum_{n \in \mathcal{N}} d^{u,(n)} = R_u,\ \forall u \in \mathcal{U}\}, \tag{11}$$

and we already separated the minimizations over the two types of variables, $\mathbf{d}$ and $\mathbf{p}$. Writing the optimum of (5) with target-rates $\mathbf{d}^{(n)}$ as $D^*(\mathbf{d}^{(n)})$, we define the partial dual problem similarly as in Section II-B by

$$D_S^* \;=\; \underset{\mathbf{d} \in \mathcal{D}}{\text{minimize}} \quad \sum_{n \in \mathcal{N}} D^*(\mathbf{d}^{(n)}) \tag{12}$$

A. An Equivalent Formulation

By the following proposition we have equivalence of Problem (10) with a more common, *coupled* problem formulation, *cf.* [2], [5],

$$\begin{aligned} &\underset{\mathbf{p}}{\text{minimize}} && \sum_{n \in \mathcal{N}, u \in \mathcal{U}} w_u \sum_{c \in \mathcal{C}} p_c^{u,(n)} && && \text{(13a)} \\ &\text{subject to} && \sum_{n \in \mathcal{N}, c \in \mathcal{C}} r_c^u(\mathbf{p}_c) \;\geq\; R_u, && \forall u \in \mathcal{U}, && \text{(13b)} \\ & && \mathbf{p}_c^{(n)} \in \mathcal{Q}_c, && \forall c \in \mathcal{C},\ \forall n \in \mathcal{N}. && \text{(13c)} \end{aligned}$$

Proposition 1: Problems (10) and (13) are equivalent in the sense that their optimal objective and solutions $\mathbf{p}^*$ are equal. Moreover, the equivalence persists if orthogonality constraints in the form of (8) are added to both problems.

The proof is omitted here due to space limitations. However, it is based on the equivalence in objective of both problems as well as feasibility of an optimal power allocation to (13) in (10) and vice versa.

B. Performance Bound

Our aim now is to bound the (weighted) energy savings that are possible by periodic scheduling additionally to that of optimal DSM. We first state our main result of this section.

Proposition 2: Assume feasibility of DSM Problem (2) with target-rates $\mathbf{R}$, optimal objective value $P^*(\mathbf{R})$ and duality-gap ζ, then the minimum objective P_S^* of cross-layer optimization Problem (10) is lower-bounded by

$$P_S^* \geq N \left(P^*(\mathbf{R}) - \zeta\right). \tag{14}$$

Proof: First we see that Problem (12) after continuous relaxation of the integer constraints is a convex optimization problem as $D^*(\mathbf{d}^{(n)}),\ n \in \mathcal{N}$, is a convex function of the scheduled rates $\mathbf{d}$ [11], and the non-negativity constraints on $\mathbf{d}$ as well as the equality constraints in (11) are linear. Next we note that the vector of optimal multipliers $\boldsymbol{\lambda}^*$ of Problem (5) is a subgradient of $D^*(\mathbf{d}^{(n)})$, *cf.* Danskin's theorem [11, p. 737]. Hence, a subgradient $\mathbf{g}$ of the objective in (12) w.r.t. $\mathbf{d}$ at average target-rates $\mathbf{d}^{(n)} = \mathbf{R}, \forall n \in \mathcal{N}$, is given by $\mathbf{g} = \boldsymbol{\lambda}^* \otimes \mathbf{1}$, where $\otimes$ denotes the Kronecker product. By definition of a subgradient [11, p.731] we have

$$\sum_{n \in \mathcal{N}} D^*(\mathbf{d}^{(n)}) \geq \sum_{n \in \mathcal{N}} D^*(\mathbf{R}) + (\mathbf{d} - \mathbf{R} \otimes \mathbf{1})^T \mathbf{g}. \tag{15}$$

Taking $\mathbf{d} \in \mathcal{D}$ we see that the second term in (15) vanishes and we have $\sum_{n \in \mathcal{N}} D^*(\mathbf{d}^{(n)}) \geq \sum_{n \in \mathcal{N}} D^*(\mathbf{R}),\ \forall\ \mathbf{d} \in \mathcal{D}$. This means that $\mathbf{d} = \mathbf{R} \otimes \mathbf{1}$ is an optimal solution to (12). Summarizing we have

$$\begin{aligned} P_S^* \geq \min_{\mathbf{d} \in \mathcal{D}} \{\sum_{n \in \mathcal{N}} D^*(\mathbf{d}^{(n)})\} &= \sum_{n \in \mathcal{N}} D^*(\mathbf{R}) && (16) \\ &= N \cdot (P^*(\mathbf{R}) - \zeta), && (17) \end{aligned}$$

where the inequality follows from weak duality [10, Ch. 5] between Problems (10) and (12), the first equality follows from optimality of $\mathbf{d} = \mathbf{R} \otimes \mathbf{1}$ in (12), and the second equality follows from (9). Another way to proof the proposition would have been to use the convexity of $D^*(\mathbf{d}^{(n)}),\ n \in \mathcal{N}$, together with arguments on the symmetry inherent in the scheduling problem (10). This symmetry is based on the fact that the per-frame subproblems (2) are identical and therefore exchangeable. □

Differently stated we have that the additional energy savings per DMT symbol due to periodic scheduling are bounded by the duality-gap ζ between the physical layer DSM Problem (2) and its partial dual (5). As mentioned in the introduction section, physical layer OFDM problems were previously enhanced by inclusion of per-carrier time-sharing [1], [5].

Assuming discrete time-shares, another conclusion from this section is therefore that the additional objective reduction per OFDM symbol by time-sharing in the weighted sum-power minimization problem is bounded by the duality-gap of the physical layer OFDM problem.

IV. THE DUALITY-GAP IN DSM

In the previous section we have seen the dependence of the performance bound for periodic scheduling on the physical layer duality-gap of Problem (2). Therefore we devote this section to investigate this issue more closely.

A. Illustrative Examples

The constraint sets of *both*, the primal and the dual problems in (2) and (5), respectively, involve integer bit-loading constraints. These integer constraints naturally imply a non-zero duality-gap for non-integer target-rates. As we excluded these by assumption due to irrelevance in practice, we will now only consider integer bit-loading solutions. In [1] curves of optimal objective values along a *line* of target-rates were investigated. The non-convexity of such a curve implies a strictly positive duality-gap for certain target-rates. The reverse conclusion does however not hold, *cf.* the geometric interpretation of dual optimization in [11, Ch. 5]. In the following we will therefore consider the dependency of the duality-gap on the target-rate. In DSL systems duality-gaps occur for instance around target-rates where users start to strictly load their bits on different subcarriers, while at rates below these points at least one subcarrier is still shared by both users. Differently, in weighted sum-rate maximization and sum-power minimization problems for OFDM systems these gaps may appear when, changing $\mathbf{R}$ or $\mathbf{P}^{\max}$, the subcarrier assignment to users changes at a primal optimal solution [1].

Figure 1(a) depicts the duality-gap ζ for any primal feasible target-rates $\mathbf{R}$ in a scenario with symmetric interference and $C = 2$ subcarriers.[2] Noteworthy, for feasible target-rates the duality-gap in this example can be as much as 93% of the corresponding optimal primal objective. Also, higher gap values seem to be located near to the boundary of the rate-region. An exception occurs at approximately equal target-rates for both users where orthogonal allocations are both, primal and dual optimal. A similar intuition can be drawn from Figure 1(b) where we depict the duality-gap in a system with orthogonality constraints (8) and 64 subcarriers.[3] The maximum duality-gap in this example amounts to approximately 3.2 % of the corresponding optimal objective value, while it is negligible in most parts of the feasible rate-region.

B. Bounds on the duality-gap

In [12] it has been shown that the duality-gap in nonconvex problems is zero if the optimal primal solution as a function of the Lagrange multipliers is continuous at all dual optima. This condition implies the convexity of the optimal weighted sum-power objective value in the target-rates. Due to this convexity property it has been argued in [6] that the duality-gap in multi-carrier systems vanishes as the number of subcarriers approaches infinity, *cf.* also [7] and similar results in the optimization literature [13, Sec. 5.6.1], [14]. Real DSL systems however use no more than a few thousand subcarriers, and it is up to now not clear what the gap is in this case, *cf.* the simple examples above and also the simulation results in [1] on the duality-gap in *orthogonal* frequency division multiplexing systems with *continuous* bit-loading.

[2]The parameters were $\Gamma = 19, H_c^{uu} = 10^{-1}, N_c^u = 10^{-14}, H_c^{ui} = 10^{-4}, i \neq u, \hat{p}_c^u = 10^{-2}, \forall c \in \mathcal{C}, \forall u \in \mathcal{U}, \mathbf{w} = [0.8, 0.2]^T$.

[3]The parameters were $P_u^{\max} = 20, \mathcal{B}_c^u = \{0, \dots, \infty\}, \hat{p}_c^u = 1, \forall u \in \mathcal{U}, \forall c \in \mathcal{C}, H_c^{11}/(\Gamma N_c^1) = c^{1.5}, H_c^{22}/(\Gamma N_c^2) = (C+1-c)^{1.5}, \forall c \in \mathcal{C}, \mathbf{w} = [1/3, 2/3]^T$ and only one-dimensional signal constellations were used for simulation complexity reasons, *cf.* also the similar setup in [1].

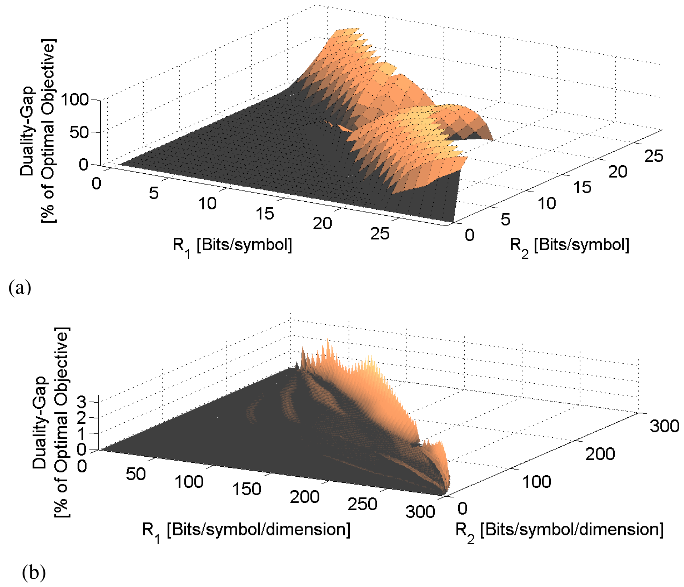

Fig. 1. Duality-gap over target-rates $\mathbf{R}$ for Problem (2), a) in a scenario with symmetric interference and $C = 2$ subcarriers, b) in a scenario with $C = 64$ subcarriers, additional sum-power constraints (4b) and orthogonality constraints (8), and using one-dimensional signal constellations only.

In [13, p. 371] the duality-gap of a general non-convex optimization problem was bounded under several assumptions. Based on this the following can be said about the duality-gap of Problem (4).

Corollary 1 (of [13, Prop. 5.26]): The duality-gap ζ between the optimal objectives of problems (4) and its partial dual after relaxation of per-user sum-power constraints can be upper-bounded by

$$\zeta \leq \max_{\substack{\{\tilde{\mathcal{C}} \mid \tilde{\mathcal{C}} \subset \mathcal{C}, \\ |\tilde{\mathcal{C}}| = (U+1)\}}} \left\{ \sum_{c \in \tilde{\mathcal{C}}, u \in \mathcal{U}} w_u \max_{\mathbf{p}_c \in \mathcal{Q}_c} \left\{ r_c^u \left(\mathbf{p}_c \right) \right\} \right\}. \quad (18)$$

The result follows from the proof to [13, Prop. 5.26] and its complete proof is hence omitted here due to space limitations.

Note that in (18) for $c \in \tilde{\mathcal{C}}, u \in \mathcal{U}$, we take the maximum feasible bit-loading for user u, implying that we can assume $p_c^i = 0, \forall i \in \mathcal{U} \setminus u$. Instead of summing over the $(U+1)$ maximum values over subcarriers, we may also upper-bound the gap by $(U+1)$ *times* the maximum value over all subcarriers, as done in [13, Prop. 5.26]. However, the assumptions

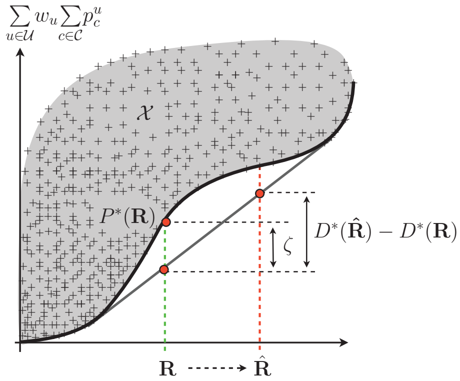

Fig. 2. Schematic illustration of the discrete set of feasible objective value - target-rate pairs $\mathcal{X}$, the duality-gap ζ according to the geometric interpretation in [11, Ch. 5], and its bound in (19).

made in the proof of [13, Prop. 5.26] do not hold for Problem (2), as integer bit-loadings and an interference channel are considered. A modified proof allows to show the following bound for the duality-gap of Problem (2), *cf.* Figure 2.

Proposition 3: The duality-gap ζ between the optimal objectives of problems (2) and its dual (5) is upper-bounded by

$$\zeta \leq D^*(\hat{\mathbf{R}}) - D^*(\mathbf{R}), \tag{19}$$

where $D^*(\hat{\mathbf{R}})$ is the optimal cost of a perturbed problem to (5) with modified target-rates $\hat{\mathbf{R}} \in \mathcal{R}^U$,

$$\hat{R}_u = R_u + \begin{cases} \Delta R_u, & \text{if } R_u > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

where

$$\Delta R_u = \max_{\{\tilde{\mathcal{C}} \ | \ \tilde{\mathcal{C}} \subset \mathcal{C}, |\tilde{\mathcal{C}}| = (U+1)\}} \Big\{ \sum_{c\in\tilde{\mathcal{C}}} \max_{\mathbf{p}_c \in \mathcal{Q}_c} \{ r_c^u(\mathbf{p}_c) \} \Big\}. \tag{21}$$

The proof is omitted here due to space limitations. It is based on an application of the Shapley-Folkman theorem [13, p. 374] to a feasible solution of a perturbed problem to (2) with increased target-rates, and the boundedness of the set of feasible per-carrier bit-loadings in (3).

Note once more that the right-hand-side in (20) may be further upper-bounded using

$$\Delta R_u \leq \ (U+1) \max_{c\in\mathcal{C}, \mathbf{p}_c \in \mathcal{Q}_c} \{ r_c^u(\mathbf{p}_c) \}. \tag{22}$$

Assuming the maximum number of bits loadable per subcarrier we make the most conservative estimate of the rate-functions' lack of convexity, *cf.* [14]. Furthermore, the computation of this bound necessitates to solve Problem (5) *twice* with different target-rate vectors. Solving the dual problem only *once* we may already apply the following relation known as "weak duality" [10, Ch. 5]. For any *primal and dual feasible* point of a not necessarily convex optimization problem it holds that

$$q\left(\tilde{\boldsymbol{\lambda}}\right) \leq L(\tilde{\mathbf{p}}, \tilde{\boldsymbol{\lambda}}) \leq \sum_{u\in\mathcal{U}} w_u \sum_{c\in\mathcal{C}} \tilde{p}_c^u, \tag{23}$$

where $q(\cdot)$ is the dual function as defined in (6) and $\tilde{\mathbf{p}}_c \in \mathcal{Q}_c, \forall c \in \mathcal{C}$, due to the assumed feasibility. The first inequality holds since $\tilde{\mathbf{p}}_c$ is not necessarily the element in $\mathcal{Q}_c$ minimizing $L_c(\cdot, \tilde{\boldsymbol{\lambda}})$, and the last inequality holds due to primal and dual feasibility of $\tilde{\mathbf{p}}$ and $\tilde{\boldsymbol{\lambda}}$, respectively. Hence a simple bound computable even during the optimization process is given by

$$\zeta \leq \Big(\sum_{u\in\mathcal{U}} w_u \sum_{c\in\mathcal{C}} \tilde{p}_c^u \Big) - q\Big(\tilde{\boldsymbol{\lambda}}\Big). \tag{24}$$

Note that this bound is less computationally complex compared to (19) and additionally bounds the sub-optimality of the primal objective after dual optimization.

Yet another bound is computable without optimizing dual Problem (5) under feasibility assumptions only.

Proposition 4: Given target-rates $\mathbf{R}$, assume that Problem (2) is feasible under the modified target-rates $\hat{\mathbf{R}}$ as defined in (20). Then the duality-gap ζ between the optimal objectives of problems (2) and its dual Problem (5) can be upper-bounded by

$$\zeta \leq \sum_{u\in\mathcal{U}} 2 \cdot \Delta R_u \cdot \max_{c\in\mathcal{C}} \{ \Delta p_c^u \}, \tag{25}$$

where

$$\Delta p_c^u = \max_{\substack{\{ \{\mathbf{p}_c, \tilde{\mathbf{p}}_c\} \ | \ \mathbf{p}_c, \tilde{\mathbf{p}}_c \in \mathcal{Q}_c, \\ r_c^u(\tilde{\mathbf{p}}_c) = r_c^u(\mathbf{p}_c) + 1, \\ r_c^i(\tilde{\mathbf{p}}_c) = r_c^i(\mathbf{p}_c), \forall i \in \mathcal{U} \setminus u\}}} \Big\{ \sum_{u\in\mathcal{U}} w_u \left(\tilde{p}_c^u - p_c^u \right) \Big\}. \tag{26}$$

The proof is omitted here due to space limitations. However, it is based on weak duality (23) and the application of Proposition 3 for modified target-rates.

We emphasize that we kept the bound general by avoiding to make any specific assumptions on optimal power allocations to (5). While loosening the bound, this in fact maximizes the set of target-rates $\mathbf{R}$ for which it is applicable. Furthermore, we expect this bound to decrease with an increasing number of subcarriers for a constant total bandwidth as it depends on the maximum number of bits loadable per subcarrier in (21).

V. Simulations

In this section we will present simulation results for all derived bounds in an upstream very high speed DSL (VDSL) scenario using equal weights in the objective. The simulation parameters were chosen according to the ETSI VDSL standard [15], with an SNR-gap $\Gamma = 12.8\,$dB, a flat spectral mask constraint at $-60\,$dBm/Hz, and two transmission bands as defined in band plan 997-M1x-M. The background noise comprised ETSI VDSL noise A added to a constant noise floor at $-140\,$dBm/Hz. The first scenario is one where all users $u \in \mathcal{U}$ are collocated at $500\,$m distance from the deployment point. This case serves solely to investigate the dependencies of the rough bound in (25) w.r.t. the number of users and the subcarrier width, *cf.* Figure 3. As can be seen the bound value increases with the number of users exponentially and eventually exceeds the largest possible value given by summation of the spectral mask constraints over all users and subcarriers. However, as expected at the end of the previous

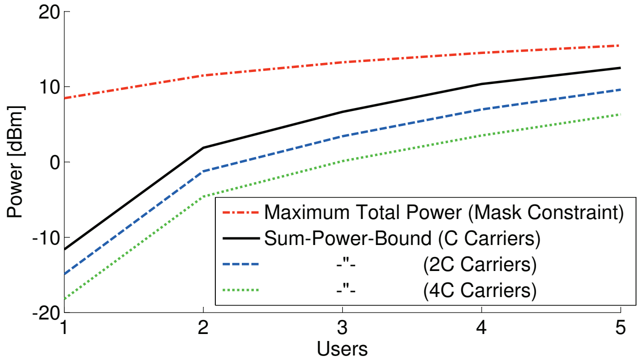

Fig. 3. Value of bound (25) on the additional energy savings by periodic scheduling over that by DSM for different numbers of subcarriers over a constant bandwidth in a DSL scenario with users collocated at 500 m.

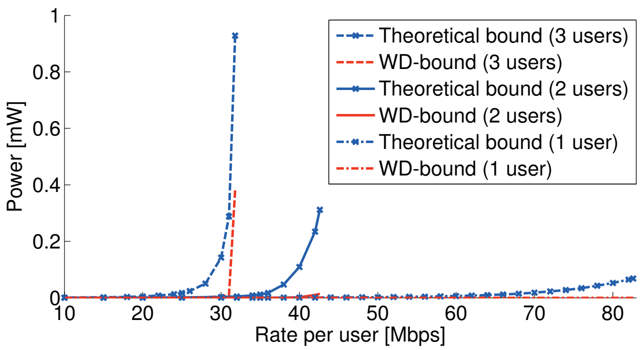

Fig. 4. Comparison between the bound in (19) and the weak duality (WD) bound in (24) on the additional energy savings by periodic scheduling in Problem (10) per symbol in a distributed DSL scenario.

section, by decreasing the subcarrier width (increasing the number of subcarriers for a constant bandwidth) the bound values also become smaller.

Next we consider distributed DSL scenarios with $U = 1, \ldots, 3$ users, located at $\{500\}$ m, $\{400, 600\}$ m, and $\{300, 500, 700\}$ m distance from the deployment point, respectively. Figure 4 compares the duality-gap bounds in (24) and (19) using these 3 scenarios and showing the dependency on the target-rates. We observe that both bounds become looser as target-rates increase, indicating a certain "loss of convexity". Moreover, the theoretical bound in (19) turned out to be less tight. Inequality (25) gives in these scenarios an upper-bound of $0.07, 1.43$, and 3.37 mW for $U = 1, \ldots, 3$, respectively. Note that these values are irrespective of the target-rates and roughly up to a factor of 5 higher than the bound in (19) from which this bound was derived. Based on the weak-duality bound we see that for all but the highest feasible target-rates the bound in (14) excludes substantial additional energy savings on top of that obtained from DSM.

VI. Conclusions

We studied the problem of joint dynamic spectrum management (DSM) and periodic rate-scheduling in time for multi-carrier systems. The additional possible energy reduction by the latter was found to be bounded by the duality-gap of the physical layer DSM problem. Furthermore, we presented bounds on this duality-gap for a finite number of subcarriers which differ in their computational complexity and requirements in model parameters. The most general bound solely assumes feasibility of increased target-rates and gets along without a dual optimization process. Furthermore, it already hints at a decrease of this bound with an increasing number of subcarriers when the total bandwidth is kept fixed. Simulation results confirm this expectation and further show for all but the highest feasible target-rates that the energy savings by periodic rate-scheduling additional to that of DSM are negligible in DSL.

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